Tfy-99.275 Lecture 2

Recapitulation of essential techniques

Linear time invariant (LTI) systems

$x(n)$: system input
$y(n)$ = R[x(n)]: system output

Linearity

$$y(n) = R[a_1 x_1(n) + a_2 x_2(n)]$$
$$y(n) = a_1 R[x_1(n)] + a_2 R[x_2(n)]$$
$$R[a_1 x_1(n) + a_2 x_2(n)] = a_1 R[x_1(n)] + a_2 R[x_2(n)]$$

Time invariance $R[x(n-m)] = z^{-m} R[x(n)]$ for all m

LTI system is completely characterized by its unit-impulse response

$$h(n) = R[d(n)]$$

input signal (sequence)

$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\delta(n-k)$$

output of a linear system with $x(n)$ as input

$$\begin{aligned} y(n) &= R[x(n)] \\ &= R\left[\sum_{k=-\infty}^{\infty} x(k)\delta(n-k)\right] \\ &= \sum_{k=-\infty}^{\infty} x(k)R[\delta(n-k)] \text{ (linearity)} \\ &= \sum_{k=-\infty}^{\infty} x(k)h(n-k) \\ (k' &= n-k) \\ &= \sum_{k'=-\infty}^{\infty} h(k')x(n-k') \end{aligned}$$

Magnitude of signals: Lp norm (p is a positive integer)

$$L_p = \left\| x(n) \right\|_p = \left(\sum_{n=-\infty}^{\infty} \left| x(n) \right|^p \right)^{\frac{1}{p}}$$

- $||x(n)||>0$ for $x(n) \neq 0$ for all n and $||x(n)||=0$ if and only if $x(n) = 0$ for all n
- $||ax(n)||=|a|*||x(n)||$ for any scalar a
- $||x(n)+y(n)|| \leq ||x(n)||+||y(n)||$ (triangle inequality)

commonly used norms

- L_1-norm: the sum of magnitudes of each sample (useful for determination of stability of linear systems)
- L_2-norm = Euclidian norm
- L_∞-norm gives peak magnitude of signal

Causality (realizable in real-time):

$$y(m) = f\,(x(n \le m), y(n < m))\,;$$
$$h(n) = 0 \ \text{ for } n < 0$$

Stability: bounded input provides bounded output:

$$|y(n)| = \left| \sum_{k=-\infty}^{\infty} h(k)x(n-k) \right|$$
$$\le \sum_{k=-\infty}^{\infty} |h(k)||x(n-k)|.$$
$$\text{let } B = \|x(n-k)\|_{\infty} \text{ then}$$
$$|y(n)| \le B \sum_{k=-\infty}^{\infty} |h(k)|$$

Stability if

$$\sum_{k=-\infty}^{\infty} |h(k)| < \infty$$
$$\|h(k)\|_1 < \infty$$

Frequency domain methods

- In physiological systems information is often more readily expressed in the frequency domain (e.g. HR in bpm, EEG freqs)
- Noise and information are often more easily separated in frequency domain than in time domain
- Efficient algorithms (e.g. fast computation of convolution in frequency domain)

Frequency domain analysis

Any signal may be presented as a series of sine waves with different amplitudes and phases

Fourier transform:

$$X(f) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j2pft}\, dt$$

$$x(t) = \frac{1}{2p} \int_{-\infty}^{\infty} X(f) \cdot e^{j2pft}\, df$$

Discrete Fourier Transform (DFT):

$$X(k) = \sum_{n=0}^{N-1} x(n) \cdot e^{-j\frac{2pnk}{N}}$$

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) \cdot e^{j\frac{2pnk}{N}}$$

Digital Filters

- general form

$$y(n) = \sum_{l=0}^{L} A_l \cdot x(n-l) + \sum_{k=1}^{K} B_k \cdot y(n-k)$$

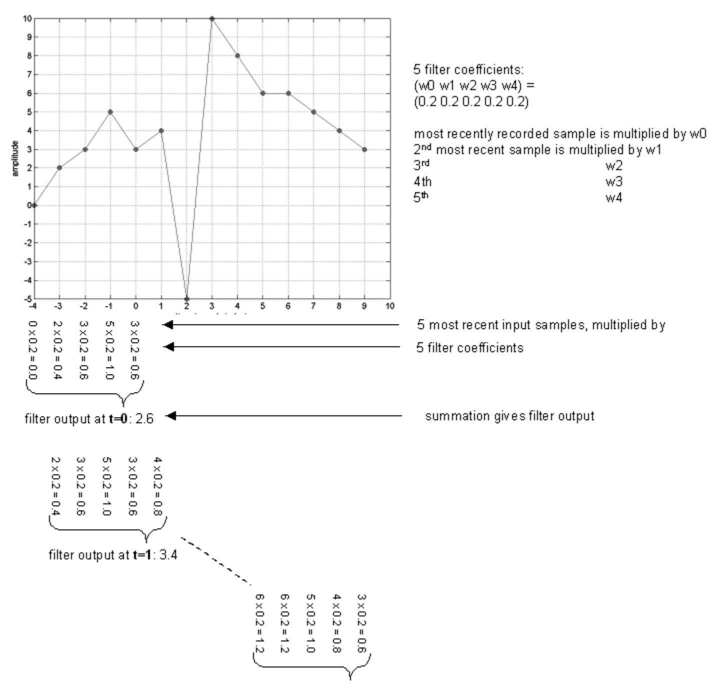
current and past inputs

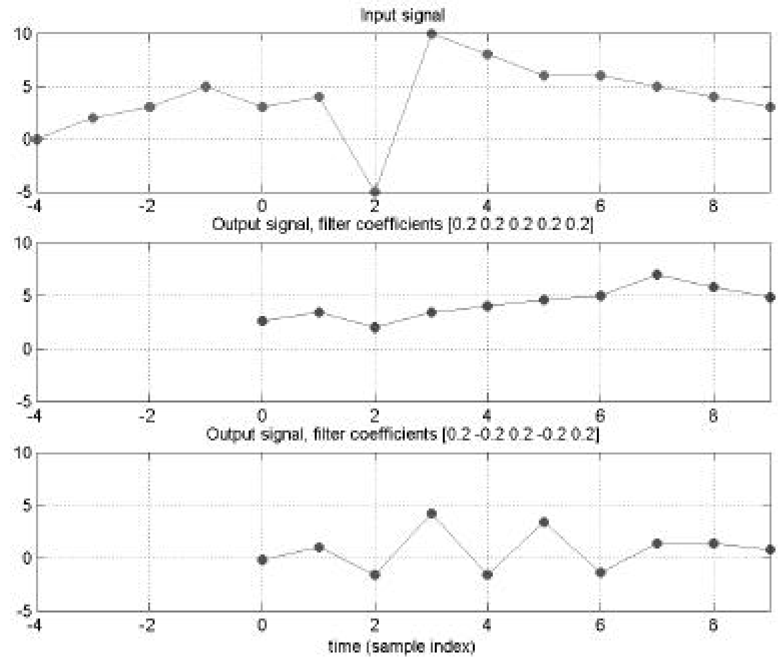
past outputs

IIR and FIR filters

- Infinite Impulse Response (IIR) filter: response is a function of both current and past inputs and past outputs. Even after the filter has stopped receiving inputs it will still generate new outputs due to the 're-use' of past outputs: h(n) is unequal to zero for all positive n (infinite range)

- Finite Impulse Response (FIR) filter: response is a function of current and past inputs only. After the filter has stopped receiving inputs it will eventually stop producing outputs: h(n) is unequal to zero in a finite range only





Frequency representations

time domain:

system output = impulse response ⊗ input

inv. Fourier ↑ ↓ Fourier transforms ↓

freq. domain

freq. output = freq. response · freq. input

also:

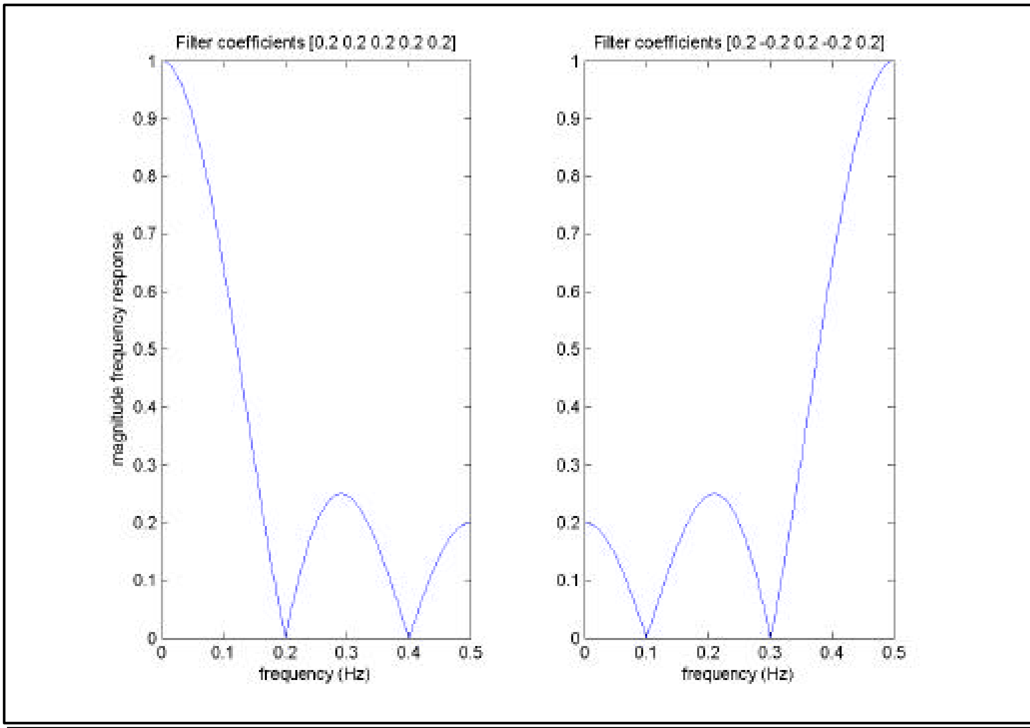
time domain

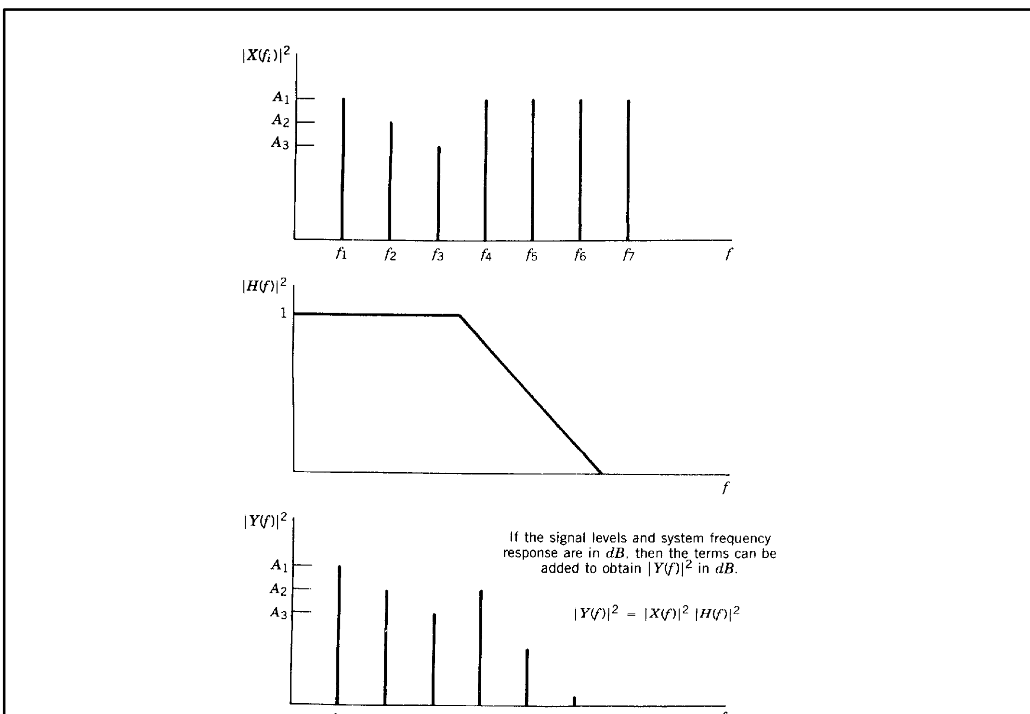
system output = impulse resp · input

inv. Fourier ↑ ↓ Fourier transforms ↓

freq. domain

freq. output = freq. response ⊗ freq. input


Filter coefficients [0.2 0.2 0.2 0.2 0.2]
Filter coefficients [0.2 -0.2 0.2 -0.2 0.2]
magnitude frequency response
frequency (Hz)
frequency (Hz)

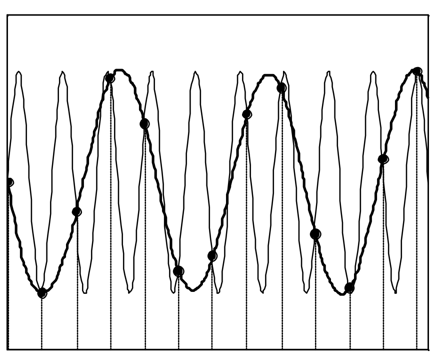

$|X(f_i)|^2$
A_1
A_2
A_3
f_1 f_2 f_3 f_4 f_5 f_6 f_7 f
$|H(f)|^2$
1
f
$|Y(f)|^2$
If the signal levels and system frequency response are in dB, then the terms can be added to obtain $|Y(f)|^2$ in dB.
$|Y(f)|^2 = |X(f)|^2\,|H(f)|^2$
f_1 f_2 f_3 f_4 f_5 f_6 f_7 f

Sampling theorem and aliasing

<u>Sampling (Shannon) theorem</u>: a continuous signal can be completely recovered from its samples if, and only if, the sampling rate is greater than twice the highest frequency of the signal (Nyqvist frequency).

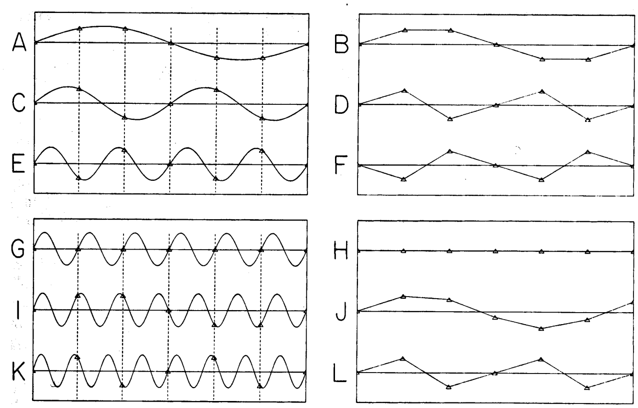
If sampling theorem is violated, <u>aliasing</u> occurs.

Aliasing in time domain



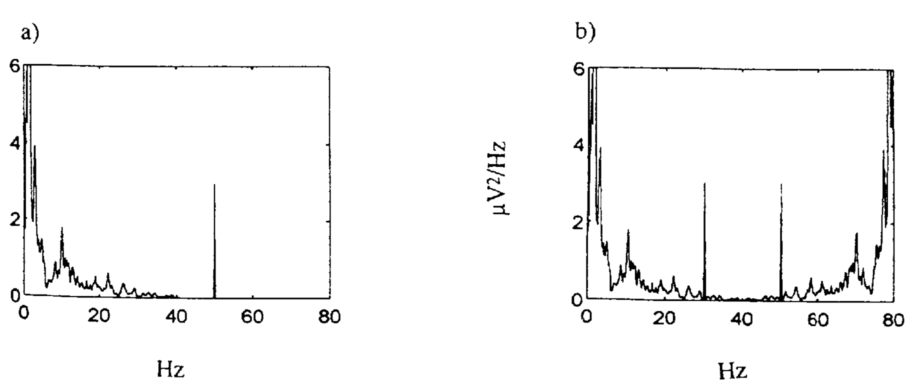
Original signal: thin line
Sampling (dashed lines) at 1.3*frequency of signal
Reconstructed signal: thick line

Aliasing in the time domain



Aliasing
Six samples per cycle of a sine wave (A) give a rather faithful representation (B). Three samples per cycle (C) still correctly represent the sine wave's frequency (D). Fewer than two samples per cycle (E–K) incorrectly represent the sine wave's frequency (F–L) as a frequency that is lower than the true frequency or even (G, H) as a constant signal.

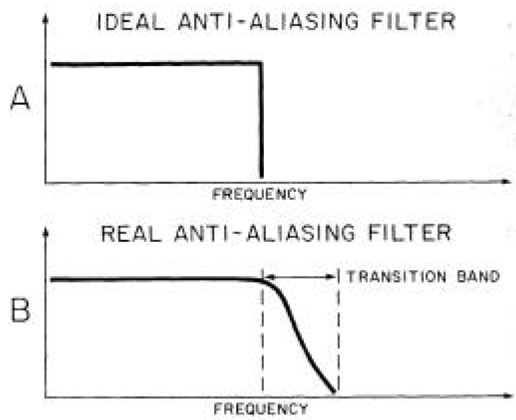
Aliasing in frequency domain



f_s=160Hz

f_s=80Hz

anti-aliasing filter



Practical Anti-aliasing Filters Require Sampling Faster Than the Theoretical Minimum Requirement
(A) Ideal anti-aliasing filter. (B) Practical anti-aliasing filter.

Z-transform

- analysis of discrete systems in frequency domain
- allows for:
  - verification of stability
  - easy estimation of frequency responses by visual inspection

Z-transform

- maps input to a complex function
- two-sided Z transform (non-causal)

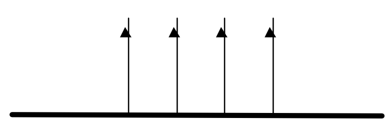
$$X(z) = \sum_{n=-\infty}^{\infty} x(nT) \cdot z^{-n}$$

- right-sided Z transform (causal)

$$X(z) = \sum_{n=0}^{\infty} x(nT) \cdot z^{-n}$$

Examples

- unit step function



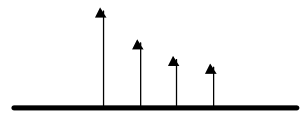
$$X(z) = \sum_{n=0}^{\infty} 1 \cdot z^{-n} = \sum_{n=0}^{\infty} 1 \cdot \left(\frac{1}{z}\right)^{n} = \frac{1}{1 - \frac{1}{z}} = \frac{z}{z-1}$$

$|z|>1$

- exponentially decreasing function

$$x(nT) = e^{-anT} \quad a > 0, n \geq 0$$

$$\text{let } x(nT) = b^n \text{ with } b = e^{-aT} < 1$$

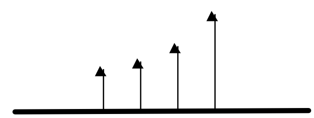


$$X(z) = \sum_{n=0}^{\infty} b^n \cdot z^{-n} = \frac{1}{1 - \frac{b}{z}} = \frac{z}{z - b}$$

- exponentially increasing function

$$x(nT) = e^{anT} \quad a > 0, n \geq 0$$

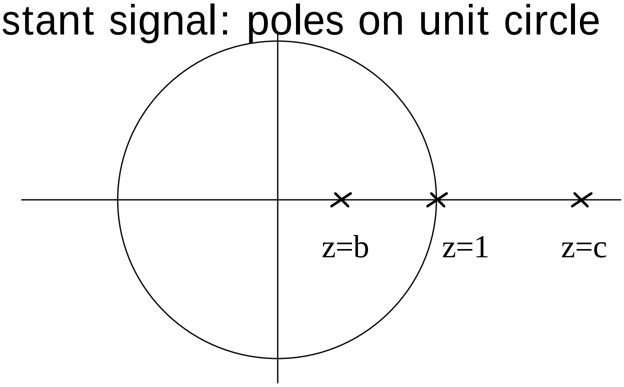
$$\text{let } x(nT) = c^n \text{ with } c = e^{aT} > 1$$



$$X(z) = \sum_{n=0}^{\infty} c^n \cdot z^{-n} = \frac{1}{1 - \frac{c}{z}} = \frac{z}{z - c}$$

location of poles

- decreasing signal: poles inside unit circle
- increasing signal: poles outside unit circle
- constant signal: poles on unit circle



some properties of the Z transform

- delay (shift) property:

$$\text{if } w(nT) = x(nT - kT) \;\text{ then }\; W(z) = X(z)\cdot z^{-k}$$

- product of convolution:

$$\text{if } w(nT) = \sum_{k=-\infty}^{\infty} x(kT)\cdot y(nT-kT) \;\text{ then }\; W(z) = X(z)\cdot Y(z)$$

Derivation of statement previous slide

$$\begin{aligned}
W(z) &= \sum_{n=-\infty}^{\infty}\left(\sum_{k=-\infty}^{\infty} x(kT)\cdot y(nT-kT)\right)\cdot z^{-n} \\
&= \sum_{k=-\infty}^{\infty} x(kT)\cdot \sum_{n=-\infty}^{\infty} y((n-k)T)\cdot z^{-n} \qquad \text{substitute } m = n - k \\
&= \sum_{k=-\infty}^{\infty} x(kT)\cdot\left(\sum_{m=-\infty}^{\infty} y(mT)\cdot z^{-m}\right)\cdot z^{-k} \\
&= X(z)\cdot Y(z)
\end{aligned}$$

input signal (sequence) to an LTI system

$$x(nT) = \sum_{k=-\infty}^{\infty} x(kT) d(nT - kT)$$

output with $x(nT)$ as input

$$y(nT) = R[x(nT)]$$

$$= R\left[\sum_{k=-\infty}^{\infty} x(kT) d(nT - kT)\right]$$

$$= \sum_{k=-\infty}^{\infty} x(kT) R[d(nT - kT)] \text{ (linearity)}$$

$$= \sum_{k=-\infty}^{\infty} x(kT) h((n-k)T)$$

$$(k' = n - k)$$

$$= \sum_{k'=-\infty}^{\infty} h(k'T) x((n-k')T)$$

Thus, we can write: $Y(z) = H(z) \cdot X(z)$

Response to a sinusoidal input $x(nT) = e^{j\omega nT}$

$$y(nT) = \sum_{k=-\infty}^{\infty} h(kT) \cdot e^{jwT(n-k)} = e^{jwnT} \sum_{k=-\infty}^{\infty} h(kT) \cdot e^{-jwkT}$$

$$= x(nT) \cdot H(z)\Big|_{z=e^{jwT}}$$

because

$$H(z) = \sum_{k=-\infty}^{\infty} h(kT) \cdot z^{-k} \text{ and thus}$$

$$H(z)\Big|_{z=e^{jwT}} = \sum_{k=-\infty}^{\infty} h(kT) \cdot e^{-jwkT}$$

response is the same sinusoid multiplied by (complex) $H(\omega)$

$H(\omega)$ is called the frequency response, it is characterized by

- its magnitude $|H(\omega)|$
- and phase $\theta(\omega)$

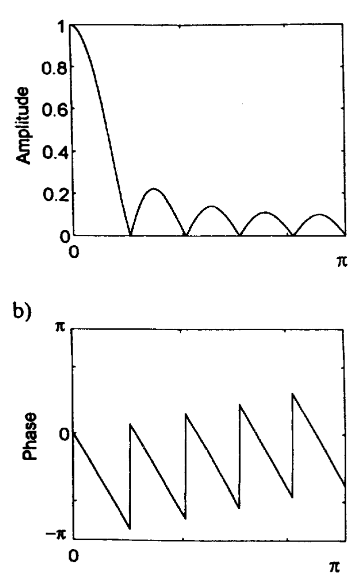
evaluate filter behavior by computing $H(z)$ for $z = e^{j\omega T}$, i.e.
along the unitary circle in the Z-plane

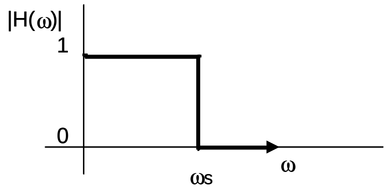
Example of a 10 points moving average filter: $y(n) = \sum_{l=0}^{9} \frac{1}{10} x(n-l)$
(commonly used in, eg. filtering of EEG recordings, with 32 or more points, and evoked potential retrieval)

$$H(z) = \sum_{k=0}^{9} \tfrac{1}{10} z^{-k}$$

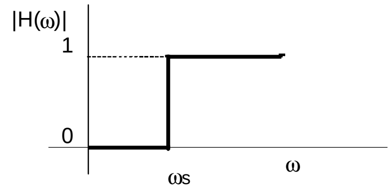
low-pass behavior

note: ω normalized to be evaluated op to π. (with a sampling freq. f_s , max freq. content = f_s /2. max ω=2π f_s /2= π f_s , or max ω = π if we normalize with respect to sampling rate

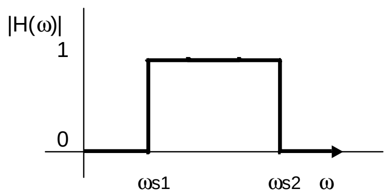




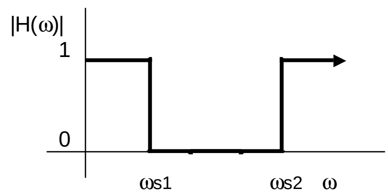
low-**pass** filter



high-pass filter



band-pass filter



band-stop filter
(notch filter if ωs1≈ωs2)

linear time-invariant systems

$$y(n) = \sum_{l=0}^{L} A_l \cdot x(n-l) - \sum_{k=1}^{K} B_k \cdot y(n-k)$$

their transfer function can be written as

$$H(z) = \frac{\sum_{l=0}^{L} A_l \cdot z^{-l}}{1 + \sum_{k=1}^{K} B_k \cdot z^{-k}}$$

if at least one $B_k \neq 0$: recursive (IIR) filter (such filters are also called autoregressive moving average (ARMA) filters)

Derivation of statement previous slide

$$y(n) = \sum_{l=0}^{L} A_l \cdot x(n-l) - \sum_{k=1}^{K} B_k \cdot y(n-k)$$

$$Y(z) = \sum_{l=0}^{L} A_l \cdot X(z) \cdot z^{-l} - \sum_{k=1}^{K} B_k \cdot Y(z) \cdot z^{-k}$$

$$Y(z)\left(1 + \sum_{k=1}^{K} B_k \cdot z^{-k}\right) = \sum_{l=0}^{L} A_l \cdot X(z) \cdot z^{-l}$$

$$Y(z) = \frac{\sum_{l=0}^{L} A_l \cdot z^{-l}}{1 + \sum_{k=1}^{K} B_k \cdot z^{-k}} X(z) = H(z) \cdot X(z)$$

by finding the roots of the numerator and denominator, the transfer function can be rewritten as:

$$H(z) = \frac{A_0 z^{K-L} \prod_{l=1}^{L} (z - z_l)}{\prod_{k=1}^{K} (z - p_k)}$$

z_l are called zeroes, p_k the poles

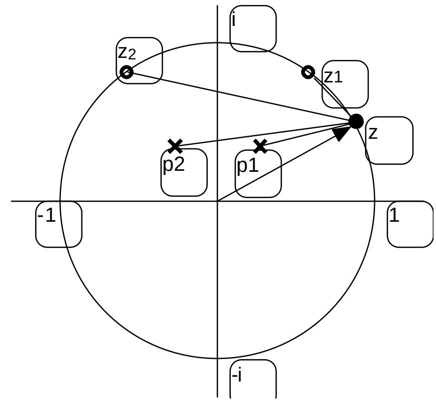
K-L zeroes on the origin and *L* elsewhere and *K* poles

stability: if all poles lie within the unitary circle, the system is stable

simple **estimation of frequency response**:
consider:

$$\vec{A}_l = (z - z_l)\Big|_{z=e^{j\varpi T}}$$

$$\vec{B}_k = (z - p_k)\Big|_{z=e^{j\varpi T}}$$



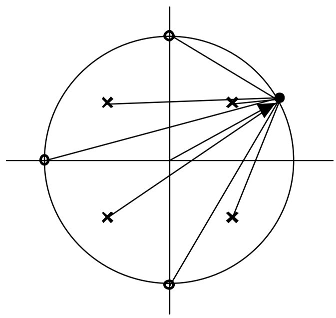
$$|H(\omega)| = \frac{A_0 \prod_{l=0}^{L} |\vec{A}_l|}{\prod_{k=1}^{K} |\vec{B}_k|}$$

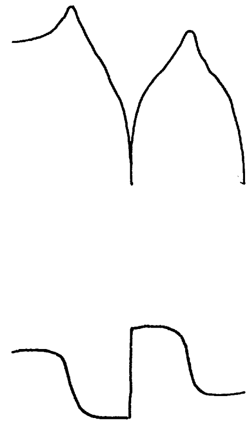
$$\theta(\omega) = \sum_{l=1}^{L} \vartheta(\vec{A}_l) - \sum_{k=1}^{K} \vartheta(\vec{B}_k) + (K-L)\omega T$$

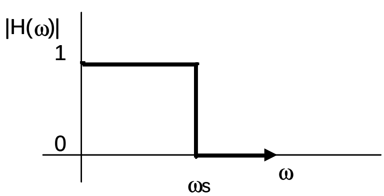
---

move around the unitary circle and use the following rules

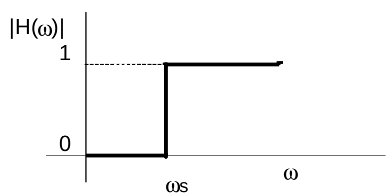
- when we are close to a zero, $|H(\omega)|$ will approach to zero and a positive phase shift will appear
- when we are close to a pole, $|H(\omega)|$ will tend to peak and a negative phase shift will appear
(note: the closer a pole to the unitary circle is, the sharper the peak will be)
- when we are near a pole-zero pair, $|H(\omega)|$ will go to zero if the zero is closer or to infinity if the pole is closer
- when we are far from a pole-zero pair, $|H(\omega)|$ can be considered 1



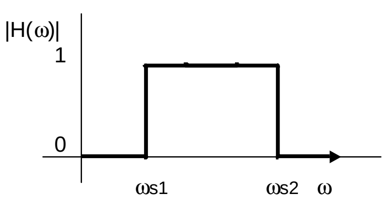




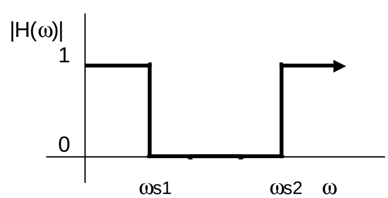
low-**pass** filter



high-pass filter



band-pass filter



band-stop filter
(notch filter if $\omega s1 \approx \omega s2$)

low-pass filtered ECG

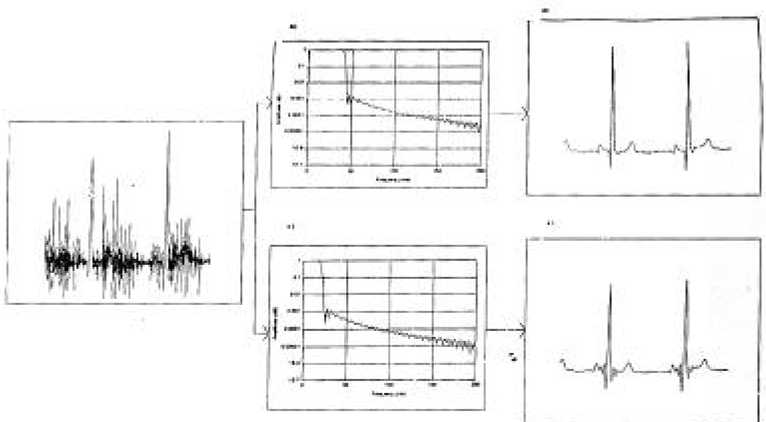


FIGURE 55.11 Effects of two different low-pass filters (b) and (c) on an ECG trace (a) corrupted by EMG noise. Both amplitude reduction and variation in the QRS width induced by too drastic low-pass filtering are evidenced.

high-pass filtered ECG

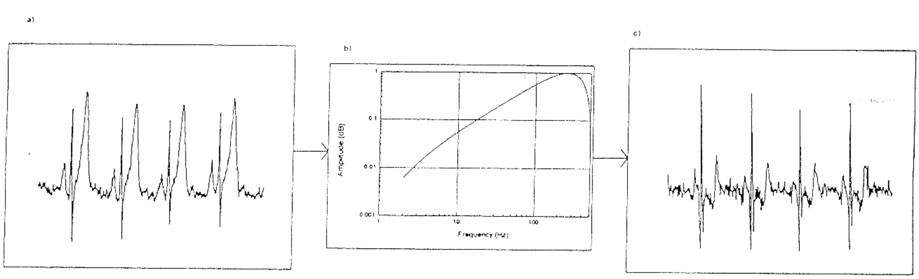


FIGURE 55.13 Effect of a derivative high-pass filter (b) on an ECG lead (a). (c) The output of the filter.

50 Hz notch-filtered ECG

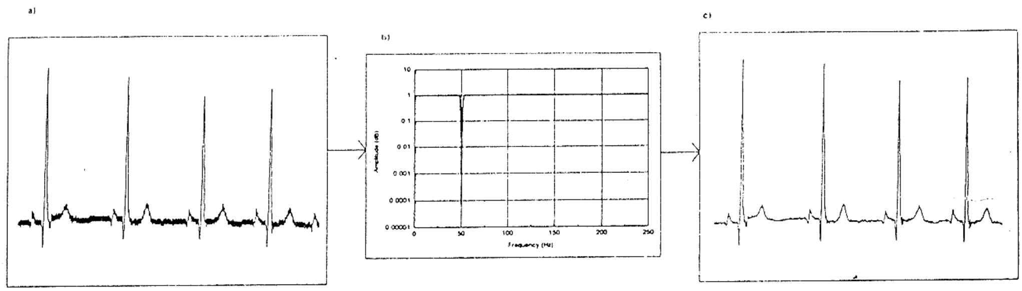
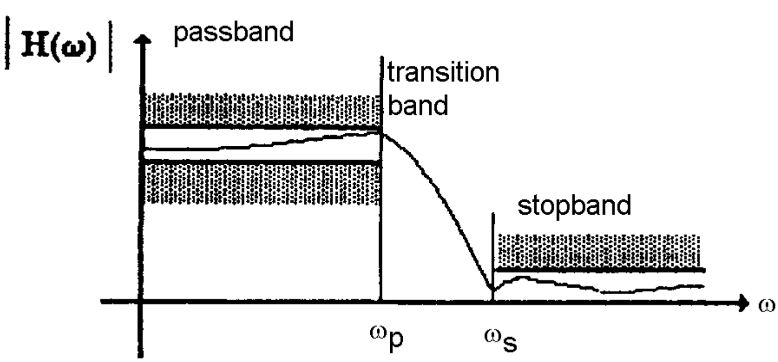


FIGURE 55.12 A 50-Hz noisy ECG signal (a); a 50-Hz rejection filter (b); a filtered signal (c)

Design of real filters



FIR vs IIR filters

+ FIR filters have linear phase shift (often important for biomedical signal interpretation)
+ FIR filters are inherently stable
+ coefficient accuracy problems that may occur for sharp IIR cut-off filters can be made less prominent for FIR filters
+ FIR filters can be realised efficiently in hardware

– FIR filters often require a much higher order than IIR filters to achieve a given level of performance

IIR design techniques

analog prototyping	use poles and zeros in continuous (Laplace) domain, obtain digital filter through frequency transformation and filter discretization	Butterworth, Chebyshev type I and II, elliptic
‘direct’ design	design in the discrete time domain by approximating a piece-wise linear magnitude response	Yule-Walker

Fourier Series Method of Designing filers

- frequency response of a digital filter is periodic with period equal to sampling frequency F

$$H(e^{j2pfT}) = \sum_{n=-\infty}^{\infty} h_d(n) \cdot e^{-2pfnT}$$

$$h_d(n) = \frac{1}{F} \int_{-F/2}^{F/2} H(e^{j2pfT}) \cdot e^{2pfnT} \cdot df$$

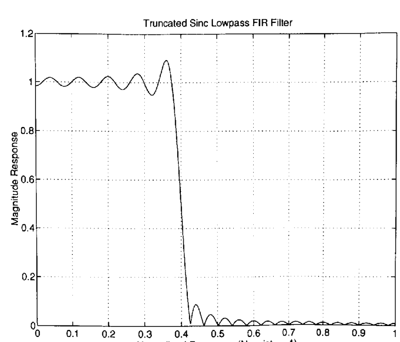
series represent desired impulse response

non-causal filter of infinite duration

- truncate infinite response and multiply result by $z^{-(N-1)/2}$

$$H(z) = z^{-(N-1)/2} \cdot \left[h_d(0) + \sum_{n=1}^{(N-1)/2} h_d(n) \cdot (z^n + z^{-n}) \right]$$

•delay factor (does not affect amplitude response)
•abrupt truncation of Fourier series leads to oscillations near discontinuities: Gibbs effect



reduction of Gibbs effect

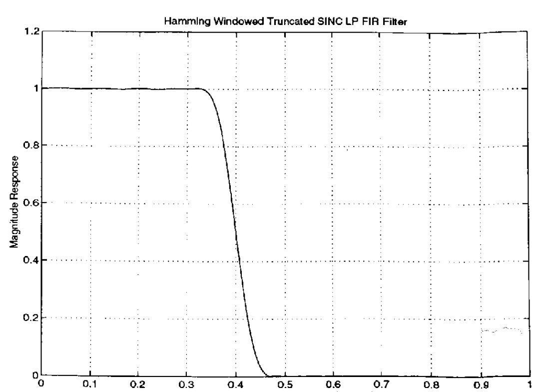
- to reduce oscillations: multiply Fourier coefficients by weights - windowing functions (preferably one whose Fourier transform has low ‘sidelobes’):
- rectangular window

$$a_R(n) = \begin{cases} 1 \ \text{for} \ |n| \le \dfrac{N-1}{2} \\ 0 \ \text{otherwise} \end{cases}$$

- Hamming window:

$$a_H(n) = \begin{cases} 0.54 + 0.46\cos\left(\dfrac{2pn}{N-1}\right) & \text{for} \ |\mathrm{n}| \le \dfrac{N-1}{2} \\ 0 & \text{otherwise} \end{cases}$$

result of applying hamming window function



Computer-aided design of filters

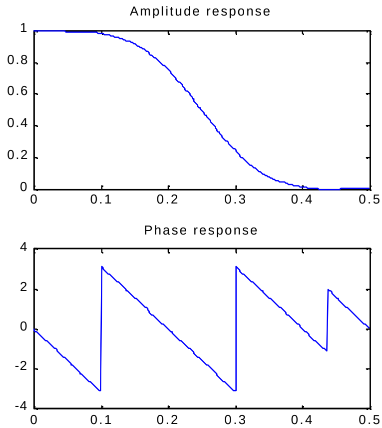
- analytic methods:
- ripples in stopband and transition bands are equal
- no minimization of number of filter coefficients

- computer aided techniques allow for weighting of the different bands, including 'don't care regions, and minimization of N

popular technique of designing FIR filters

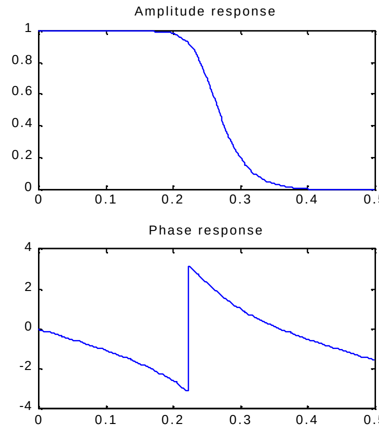
- iterative algorithm that minimises of the maximum error (minimax algorithm) between the desired frequency response and the actual response (Parks-McClellan/Remez algorithm)

10th order FIR filter



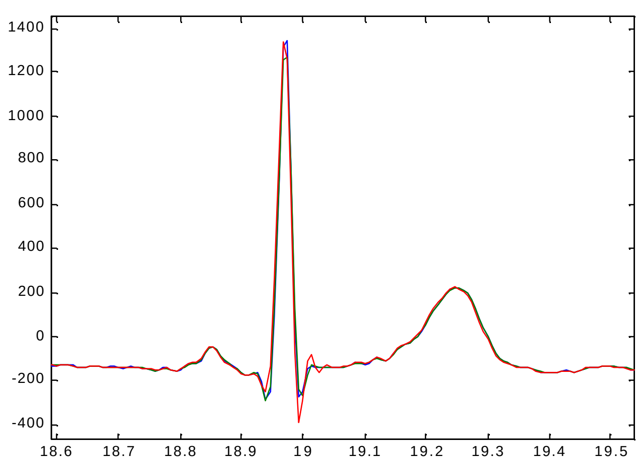
Wider transient band
Linear phase
Greater delay

5th order IIR (Butterworth) filter



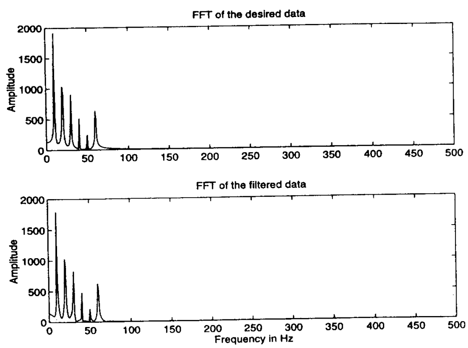
Steeper amplitude response
Nonlinear phase

EKG filtered with IIR and FIR filters

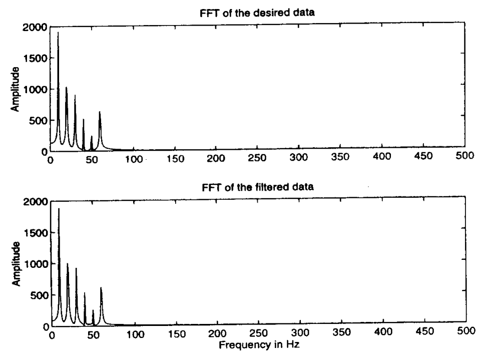


Sampling rate 150Hz
Filters from previous slide
Blue - original ECG
Green - FIR
Red - IIR

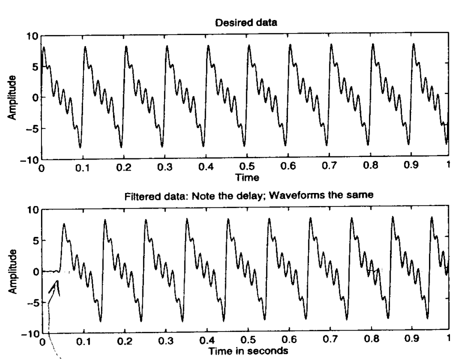
COMPARISON BETWEEN THE DESIRED SIGNAL AND THE NOISY SIGNAL FILTERED WITH THE AID OF AN FIR FILTER



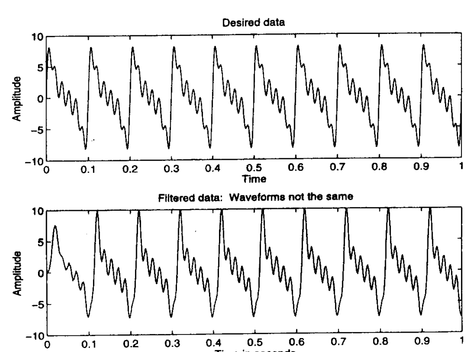
COMPARISON BETWEEN THE DESIRED SIGNAL AND THE NOISY SIGNAL FILTERED WITH THE AID OF AN IIR FILTER



COMPARISON BETWEEN THE DESIRED SIGNAL AND THE NOISY SIGNAL FILTERED WITH THE AID OF AN FIR FILTER



COMPARISON BETWEEN THE DESIRED SIGNAL AND THE NOISY SIGNAL FILTERED WITH THE AID OF AN IIR FILTER



Linear vs non-linear

The application field, biomedicine, usually deals with systems (humans or animals) that do not even come close to 'being linear', e.g.,

- noise is dependent on the signal (multiplicative noise)
- signal statistics are non-stationary
- noise or signals have non-Gaussian distributions
- human perception and information processing is highly non-linear
- .....

So, what good are linear processing techniques anyway?

Why linear methods?

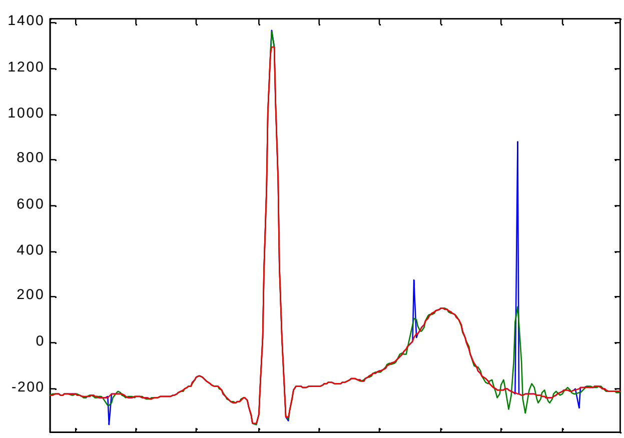
- physiological signals/systems often can be viewed upon as having linear as well as non-linear components, therefore linear methods may well be used as a first (and sometimes very adequate) attempt to describe systems and signals
- many non-linear methods require prior information concerning possible non-linearities (that may not be available)
- linear methods are more 'understandable' in their behaviour than non-linear ones

- Non-linear methods may be superior over linear methods, however in the 'real world', linear techniques are still the most common

Simple non-linear example: Median filter

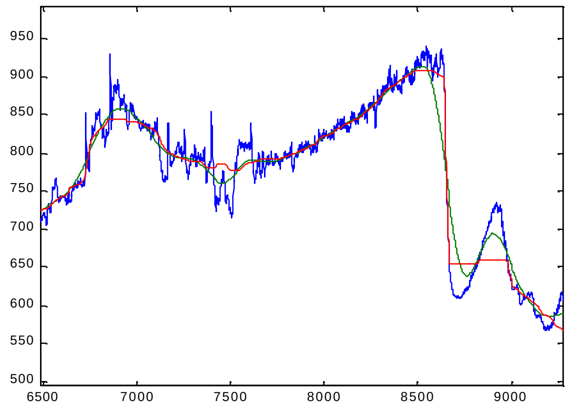
- input sequence: $[x(n-M), x(n-M+1), \ldots, x(n), \ldots, x(n+M)]$
- reorder so that samples are arranged in ascending order of magnitude, $[x(n-M), x(n-M+1), \ldots, x(n), \ldots, x(n+M)]$
- output of a median filter with 2M+1 input samples is the sample with index M+1
- median filter:
  - good at removing sharp short lasting artifacts (shot noise)
  - good at restoring step changes / edges
  - 'problem': response in frequency domain depends on input (non-linearity)

'Best case' for median filtering: shot noise



Blue: original EKG with shot-noise
Green: FIR (N=151) filtered EKG
Red: median (N=3) filtered EKG

Median filtering and step change in signal



Blue: original RRI
Green: FIR (N=300) filtered RRI
Red: median (N=300) filtered RRI

Filter design in Matlab

- FIR: fir1, fir2, firls, remez
- IIR: butter, checy1, cheby2, ellip, yulewalk
- median: medfilt1
- frequency response: freqz

Frequency domain analysis

Any signal may be presented as a series of sine waves with different amplitudes and phases

Fourier transform:

$$X(f) = \int_{-\infty}^{\infty} x(t) \cdot e^{-j2pft} dt$$

$$x(t) = \frac{1}{2p} \int_{-\infty}^{\infty} X(f) \cdot e^{j2pft} df$$

<u>Discrete Fourier Transform (DFT):</u>

$$X(k) = \sum_{n=0}^{N-1} x(n) \cdot e^{-j\frac{2pnk}{N}}$$

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) \cdot e^{j\frac{2pnk}{N}}$$

DFT

- assumption: the N samples used for calculations describe one period of a fictitious signal.
- how well the frequency spectrum of this fictive signal corresponds to that of the original signal depends on the approximations involved [aliasing, leakage effects, picket-fence effect]

- note the difference between DFT and filtering:
- DFT: multiplication between signal and weighted function in time domain
- Filtering: convolution of input signal with impulse response

- frequencies (F is sampling frequency):

$$f_k = k \cdot \frac{F}{N} k = 0,1,\ldots, N-1$$

- frequency spectrum is periodic with sampling frequency

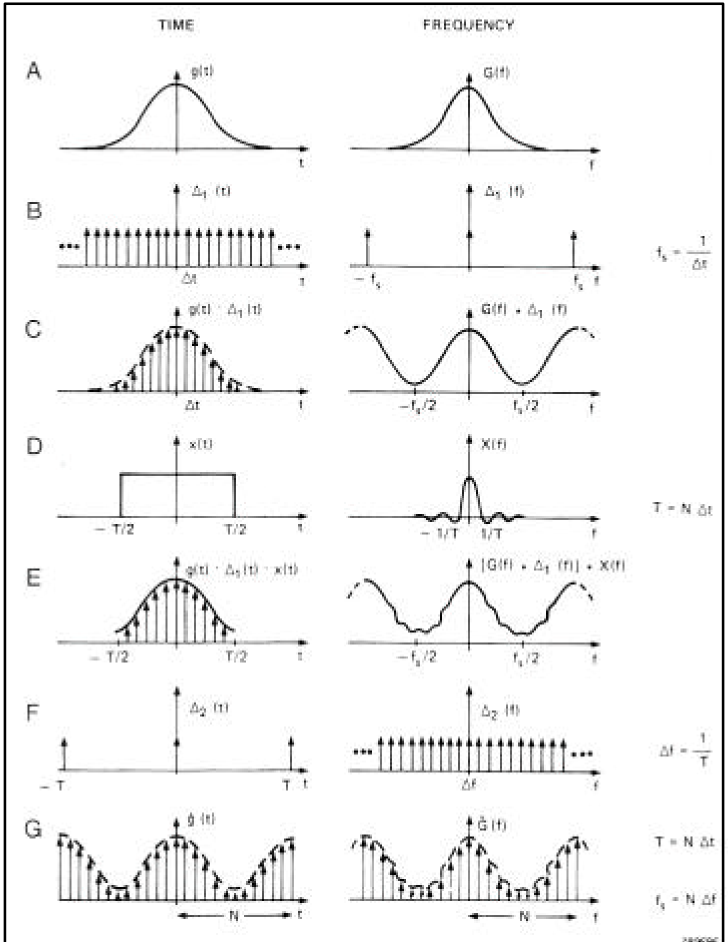
$$f_{discrete} = f_{continuous} \pm n \cdot F$$

- Shannon (sampling) theorem: aliasing if the signal is not band-limited to less than half the sampling frequency

- In physiological signals information has often a special distribution in frequency domain (e.g. EEG, HRV) or the change in frequency is related in changes in frequency content (e.g. EEG, HRV, fatigue in EMG)

Steps in DFT

1. *time sampling*
   Multiply continuous signal with sampling function (series of pulses whose Fourier transform is another series of pulses with a distance of *F* Hz) = convolution of Fourier transformed signal with pulses → periodicity in frequency domain
2. *time limitation*
   multiply signal in time domain by window function (e.g. rectangular, whose Fourier transform is a sinc function) = convolution of periodic spectrum with Fourier transform of window function → 'leakage' to neighboring frequencies
3. *time convolution due to frequency sampling*
   sample frequency spectrum into N samples within one period = convolution in time domain with pulses T sec. apart → periodicity in time domain



gure 1.69
erivation of the Discrete Fourier Transform From the Integral Transform
'rom Thrane N: The discrete Fourier transform and FFT analyzers, *Bruel & Kjaer Tech Re*
979;No. 1. Reproduced by permission.)

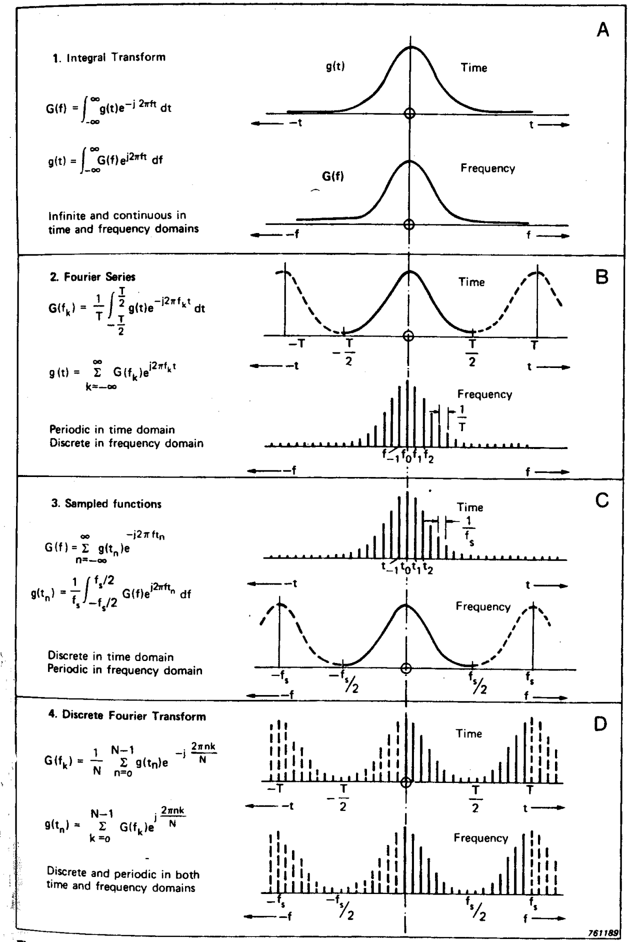
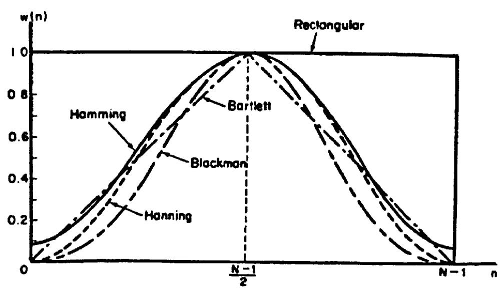


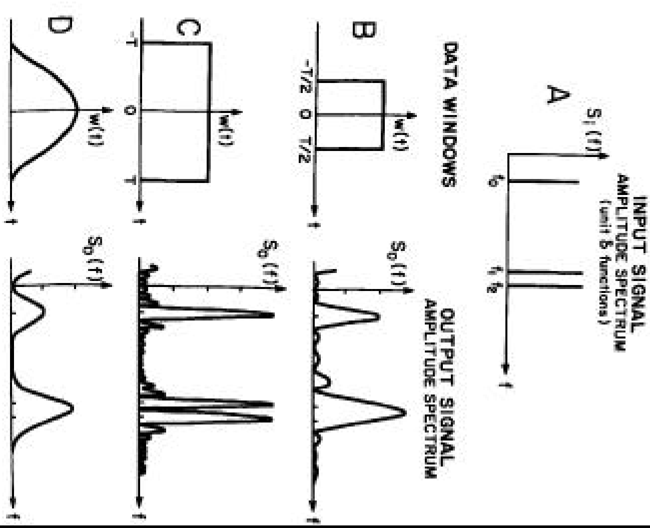
Figure 1.68
Various Forms of the Fourier Transform
(From Thrane N: The discrete Fourier transform and FFT analyzers. *Bruel & Kjaer Tech Rev* 1979;No. 1. Reproduced by permission.)

reduce leakage effects

- methods to reduce leakage effects: use windows that have low sidelobes (examples: Bartlett, Hanning, Hamming, Blackman windows), and use many samples to obtain a narrow main lobe



Spectral resolution and record length



Fast Fourier Transform (FFT)

- in practical applications often DFTs have to be computed in real-time on contiguous sets of input samples: for N frequency output points N^2 complex multiplies and $N^2 - N$ complex additions are needed
- by making smart use of relations between different Fourier coefficients FFT can be reduced to $(N/2)^2\log(N)$ multiplies and $N^2\log(N)$ additions
- note: this technique requires the number of samples to have certain restrictions, preferably a power of 2

Short-Term Fourier Transform (STFT)

- Fourier techniques assume stationarity of the signal, most signals (also biomedical ones) are not stationary $\rightarrow$ divide signal into ‘stationary’ segments
- the duration of each segment have to be determined by using *a priori* information or by examining the signal’s local characteristics

$$STFT\,(f, t) = FT\,(x(t) \cdot w(t - t))$$

- perform Fourier transform on a window of data that slides along the time axis $\rightarrow$ time-frequency function that describes the frequency distribution near time τ.

- Time-frequency resolution trade-off: shorter segments (narrower windows) give better resolution in the time domain but increase the window width in the frequency domain and thus lead to poorer frequency resolution.
  - Wavelet transforms can be used to deal with this problem better
- In highly non-stationary signals usually equal length windows are used (e.g., for speech signals windows in the order of 10 to 20 ms)
- For many other signals, variable duration windows are used (e.g., EEG signals windows in the order of 5 to 30 s)

Spectral Estimation and Analysis

The Power Spectral Density function (PSD) is defined as the Fourier transform of the autocorrelation function

$$PSD[x(t)] = S_{xx}(w) = FT[f_{xx}(t)] = \int_{-\infty}^{\infty} f_{xx}(t) \cdot e^{-jwt} dt$$

with autocorrelation function (acf)

$$f_{xx}(t) = E\{x(t)x(t-t)\} = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)x(t-t)dt$$

in the discrete case it is the z-transform of the acf at $z = e^{jwT}$:

$$\Phi_{xx}(z) = \sum_{m=-\infty}^{\infty} f_{xx}(m) \cdot z^{-m}$$

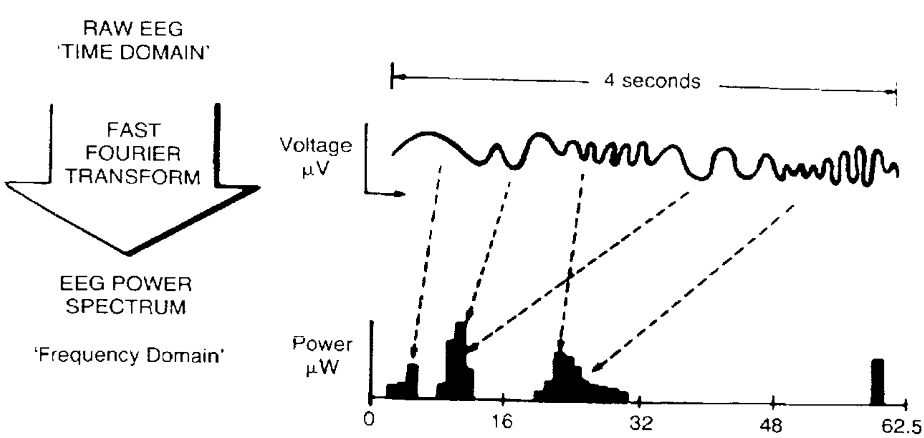
$$f_{xx}(m) = E\{x(n)x(n-m)\} = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} x(n)x(n-m)$$

- note: since the acf is even, the PSD is purely real
- the PSD is often used to describe stochastic signals, an important application area is that of EEG analysis.
- The cross-power spectral density function (CPSD) is used to study the relationship between two processes (e.g., when two sides of the brain are investigated with EEG recordings):

$$CPSD[x(t), y(t)] = S_{xy}(w) = FT[f_{xy}(t)] = \left|S_{xy}(w)\right| e^{jq_{sy}(w)}$$

- Cross-correlation function is not even $\rightarrow$ CPSD is complex!

EEG in frequency domain



Cross-correlation analysis

$$f_{xy}(m) = E\left[x(n)\,y(n+m)\right]$$

if $y(t)$ is system output and the system impulse response is $h(k)$:

$$f_{xy}(m) = \sum_{j=-\infty}^{\infty} h(j) \cdot f_{xx}(m-j)$$

It can be shown that

$$\left|S_{xy}(w)\right|^2 \le S_{xx}(w) \cdot S_{yy}(w)$$

The coherence function

$$g_{xy}^2 = \frac{\left|S_{xy}(w)\right|^2}{S_{xx}(w) \cdot S_{yy}(w)} \le 1$$

is used in a variety of biomedical applications (e.g., to investigate brain assymetry, to assess cardiac baroreflex)

Methods to estimate PSD

Non-parametric and parametric methods

<u>Non-parametric methods</u>:

<u>Blackman-Tukey</u>: calculate PSD directly from its definition, but use finite integration time and a (biased) estimate of the true correlation function

$$\hat{f}_{xx}(m) = \frac{1}{N} \sum_{i=0}^{N-1-|m|} x(i) \cdot x(m+i)$$

use FFT of this correlation function to calculate PSD

Periodogram (Welch method)

It can be shown that

$$S_{xx}(w) = \lim_{T\to\infty} E\left[\frac{1}{2T}\left|\int_{-T}^{T} x(t)\cdot e^{-jwt}dt\right|^2\right]$$

discrete form

$$\hat{S}_{xx}(w) = \frac{T}{N}\left|DFT\{x(m)\}\right|^2$$

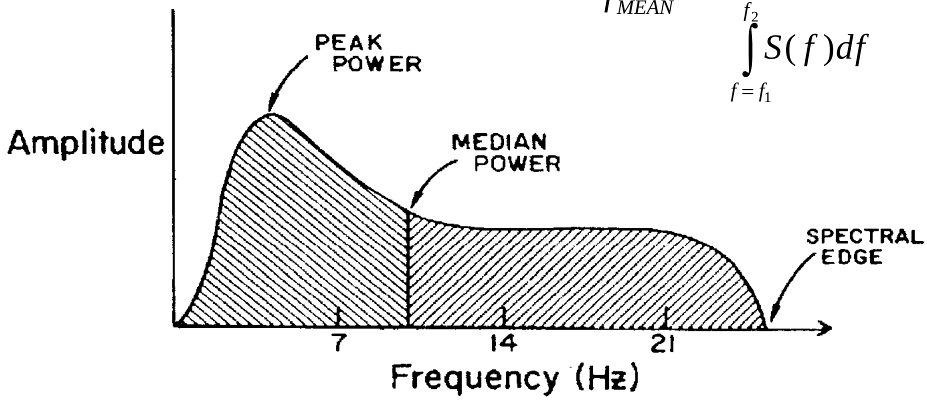
- can be implemented efficiently using FFT and no need to calculate acf
- rather than using one segment of *N* samples, the (weighted) average of periodograms of shorter (overlapping) subsegments can be used to generate a smoother PSD

Quantification of spectral analysis

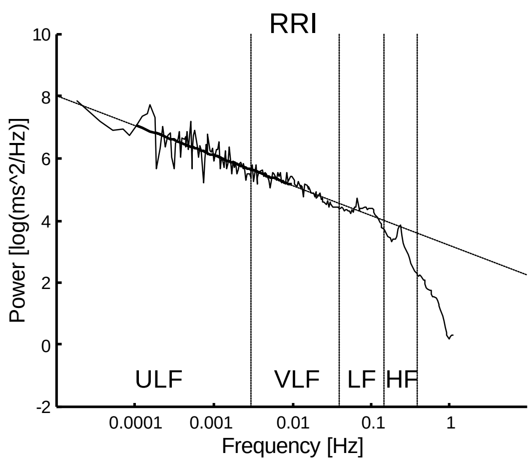
- After spectral estimation the features of the spectrum (cross-spectrum, transfer function, etc) need to be quantified
- Quantification is based on
  - spectral power (or transfer gain) in certain frequency bands (absolute or relative power, power ratios): EEG, heart rate variability (HRV)
  - frequency distribution (peak power frequency, mean or median frequency, spectral edge frequency): EMG, EEG, HRV
  - other features, e.g. log-log slope in HRV

median frequency: $$\int_{f=f_1}^{f_{MED}} S(f)\,df = \int_{f=f_{MED}}^{f_2} S(f)\,df$$

mean frequency: $$f_{MEAN} = \frac{\int_{f=f_1}^{f_2} fS(f)\,df}{\int_{f=f_1}^{f_2} S(f)\,df}$$



Wide-band spectral analysis of HRV



HRV spectrum over 24h

Plot in log-log scale

Fit line between 0.0001-0.01 Hz

Spectrogram analysis of HRV during tilt

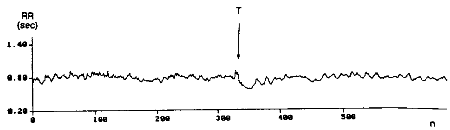


Fig. 2. RR interval tachogram in a subject passing from clinostatic to orthostatic position. The marker T indicates the beginning of transition.

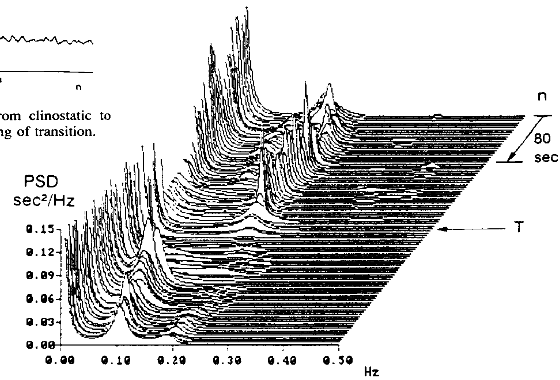
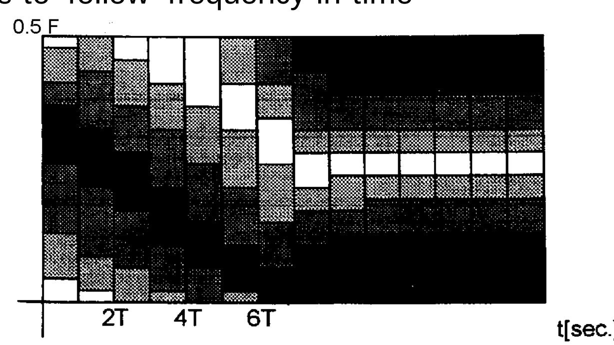


Fig. 4. CSA obtained for $w = 0.98$, relative to the tachogram shown in Fig. 2 (see also text).

Spectrogram

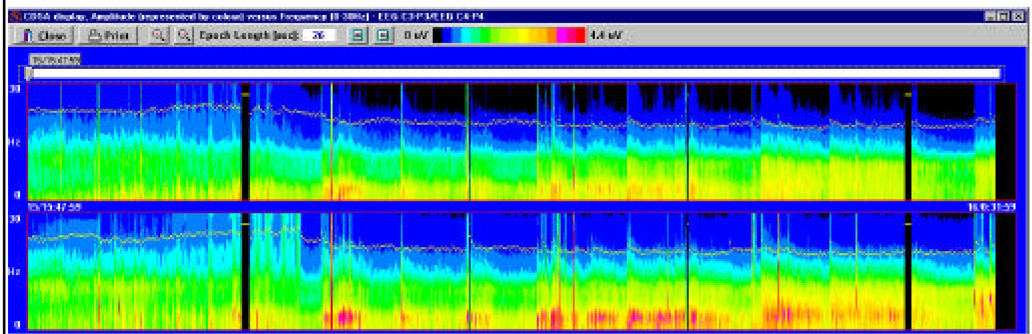
use colours to 'follow' frequency in time



here: black is maximum power, white is minimum.

non-stationary (decreasing peak frequency) from 0 to 7T, after that stationarity with one low-freq and one high-freq peak

CDSA of EEG during intensive care



Deeply sedated → Light sedation

Summary of non-parametric methods

- non-parametric methods are widely diffused:
  - simple applicability, computational speed, and direct interpretation of results
- disadvantages:
  - estimations are used that are not statistically consistent (and that should be corrected)
  - data are ‘windowed’ (leakage) → limited frequency resolution
  - assumptions about data outside the window are not always realistic

Higher order spectral analysis

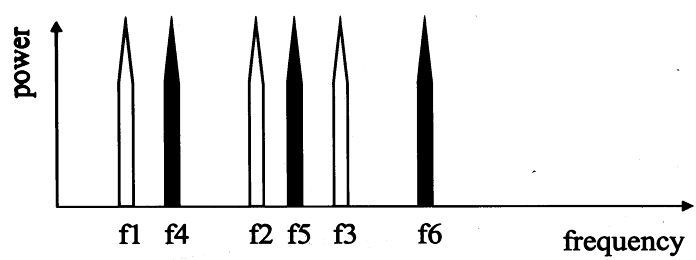
- In power spectral analysis (linear analysis, gaussian signals) only magnitude of frequencies can be seen but NO phase relations between frequencies.
- Higher order spectral analysis may reveal more complex (non-linear) relationships. Bispectral analysis reveal couplings between frequencies.
- Bispectrum of a stationary process $\{X(k)\}$ is defined as

$$B(w_1, w_2) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} C(m,n) \cdot e^{-j(w_1 m + w_2 n)}$$

with

$$C(m,n) = E\{X(k)X(k+m)X(k+n)\}$$

- for a Gaussian process $C(m,n) = 0$ for each m and n

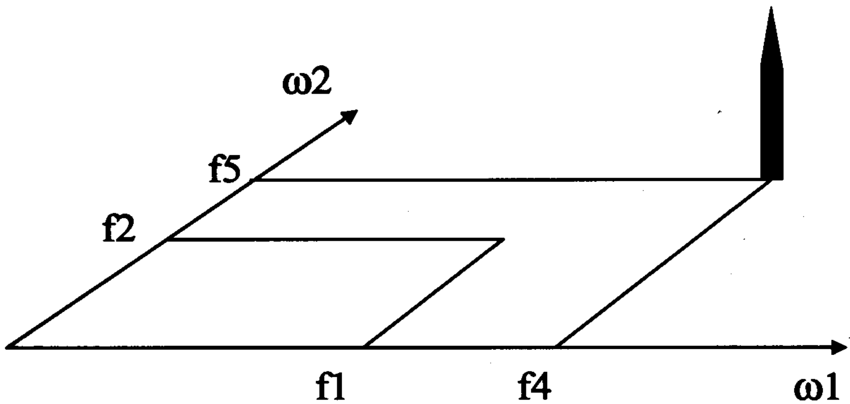


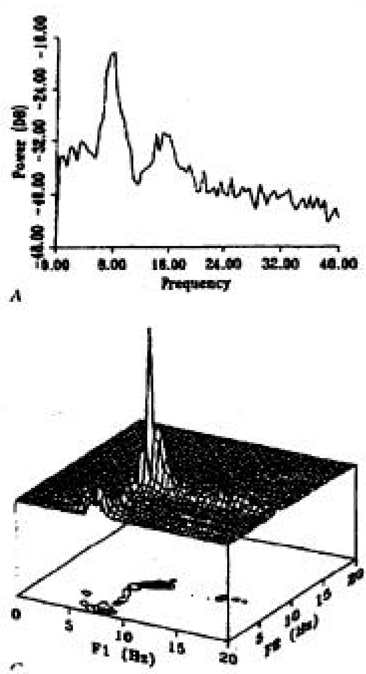
a: f1+f2 = f3, but f3 is independent from f1 and f2

b: f4+f5 = f6, but f6 is a result of the coupling between f4 and f5

a power spectrum cannot discriminate whether situation **a** or **b** exists, bispectrum can!

magnitude of bispectrum $|B(\omega 1,\omega 2)|$





EEG

power spectrum

bispectrum shows
coupling of frequencies